N=4 supersymmetric mechanics with nonlinear chiral supermultiplet

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Abstract

We construct $N = 4$ supersymmetric mechanics using the $N = 4$ nonlinear chiral supermultiplet. The two bosonic degrees of freedom of this supermultiplet parameterize the sphere S^2 and go into the bosonic components of the standard chiral multiplet when the radius of the sphere goes to infinity. We construct the most general action and demonstrate that the nonlinearity of the supermultiplet results in the deformation of the connection, which couples the fermionic degrees of freedom with the background, and of the bosonic potential. Also a non-zero magnetic field could appear in the system.

Introduction.

Supersymmetric mechanics with $N \geq 4$ supersymmetry has plenty of features which make it interesting, not only due to its relation with higher dimensional theories, but also as an independent theory. For instance, $N = 4$ superconformal symmetry in $d = 1$ is related with the one-parameter supergroup $D(2, 1; \alpha)$ [1], while in higher dimensions the superconformal group does not contain any parameters. Another interesting feature is the diversity of off-shell multiplets of $N \geq 4$ supersymmetry in $d = 1$. For example, for $N = 4, d = 1$ supersymmetry there are five off-shell linear finite supermultiplets [2] and two *nonlinear* ones [3]. One of them is a one-dimensional analog of the $N = 2, d = 4$ nonlinear multiplet [4]. The second one (called in [3] nonlinear chiral supermultiplet - NCS) seems to have no known higher-dimensional analogs. It includes, as a limiting case, the standard chiral supermultiplet and has the same component content as the latter.

The idea of the nonlinear chiral supermultiplets comes about as follows. If the two bosonic superfields $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ parameterize the two dimensional sphere $SU(2)/U(1)$ instead of flat space, then they transform under $SU(2)/U(1)$ generators with the parameters $a, \bar{a}$ as

$$\delta\mathcal{Z} = a + \bar{a}\mathcal{Z}^2, \quad \delta\overline{\mathcal{Z}} = \bar{a} + a\overline{\mathcal{Z}}^2, \tag{1}$$

With respect to the same group $SU(2)$, the $N = 4$ covariant derivatives could form a doublet[1]

$$\delta D_i = -a\overline{D}_i, \quad \delta\overline{D}_i = \bar{a}D_i\ . \tag{2}$$

Here, the covariant spinor derivatives $D^i, \overline{D}_j$ are defined in the superspace $\mathbb{R}^{(1|4)}$ by

$$D^i = \frac{\partial}{\partial\theta_i} + i\bar{\theta}^i\partial_t\ , \ \overline{D}_i = \frac{\partial}{\partial\bar{\theta}^i} + i\theta_i\partial_t\ , \quad \{D^i, \overline{D}_j\} = 2i\delta^i_j\partial_t. \tag{3}$$

One may immediately check that the ordinary chirality conditions $D_i\mathcal{Z} = 0, \quad \overline{D}_i\overline{\mathcal{Z}} = 0$ are not invariant with respect to (1), (2) and they should be replaced, if we wish to keep $SU(2)$ symmetry. It is rather easy to guess the proper $SU(2)$ invariant constraints

$$D_i\mathcal{Z} = -\alpha\mathcal{Z}\overline{D}_i\mathcal{Z}, \quad \overline{D}_i\overline{\mathcal{Z}} = \alpha\overline{\mathcal{Z}}D_i\overline{\mathcal{Z}}\ , \qquad \alpha = \text{const}. \tag{4}$$

So, using the constraints (4), we restore the $SU(2)$ invariance, but the price for this is just the nonlinearity of the constraints. Let us stress that $N = 4, d = 1$ supersymmetry is the minimal one where the constraints

[1] In the trivial situation, when the covariant derivatives do not transform under $SU(2)$, this $SU(2)$ has nothing to do with the R-symmetry of the fermionic sector of the theory. In this case nothing interesting is happening.

(4) may appear, because the covariant derivatives (and the supercharges) form a doublet of $SU(2)$ which cannot be real.

In this paper we construct $N = 4$ supersymmetric mechanics with two bosonic and four fermionic degrees of freedom starting from the most general action for NCS in $N = 4$ superspace (a preliminary step in the construction of such system was done in [3]). We find that the configuration space of the system is defined by a connection different from the symmetric connection of the base space, in contrast with standard $N = 4$ sypersymmetric mechanics with linear chiral supermultiplet. Also, the superpotential terms give rise to an interaction with the magnetic field which preserves $N = 4$ supersymmetry. The potential term is also modified as compared to $N = 4$ mechanics with linear chiral supermutiplet. In a special limit, our action admits a reduction to the well known case of $N = 4$ mechanics (see, e.g. [5, 6]).

Superfields formulation.

As we already mentioned, the $N = 4, d = 1$ NCS involves one complex scalar bosonic superfield $\mathcal{Z}$ obeying the constraints (4). If the real parameter $\alpha \neq 0$, it is always possible to pass to $\alpha = 1$ by a redefinition of the superfields $\mathcal{Z}, \overline{\mathcal{Z}}$. So, it has only two essential values $\alpha = 1$ and $\alpha = 0$. The latter case corresponds to the standard $N = 4, d = 1$ chiral supermultiplet. Now one can write the most general $N = 4$ supersymmetric Lagrangian in $N = 4$ superspace[2]

$$S = \int dt d^2\theta d^2\bar\theta \; K(\mathcal{Z}, \overline{\mathcal{Z}}) + \int dt d^2\bar\theta \; F(\mathcal{Z}) + \int dt d^2\theta \; \overline{F}(\overline{\mathcal{Z}}) \; . \tag{5}$$

Here $K(\mathcal{Z}, \overline{\mathcal{Z}})$ is an arbitrary function of the superfields $\mathcal{Z}$ and $\overline{\mathcal{Z}}$, while $F(\mathcal{Z})$ and $\overline{F}(\overline{\mathcal{Z}})$ are arbitrary holomorphic functions depending only on $\mathcal{Z}$ and $\overline{\mathcal{Z}}$, respectively. Let us stress that our superfields $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ obey the nonlinear variant of chirality conditions (4), but nevertheless the last two terms in the action S (5) are still invariant with respect to the full $N = 4$ supersymmetry. Indeed, the supersymmetry transformations of the integrand of, for example, the second term in (5) read

$$\delta F(\mathcal{Z}) = -\epsilon^i D_i F(\mathcal{Z}) + 2i\epsilon^i \bar\theta_i \dot F(\mathcal{Z}) - \bar\epsilon_i \overline{D}^i F(\mathcal{Z}) + 2i\bar\epsilon_i \theta^i \dot F(\mathcal{Z}) \; . \tag{6}$$

Using the constraints (4) the first term in the r.h.s. of (6) may be rewritten as

$$-\epsilon^i D_i F = -\epsilon^i F_{\mathcal{Z}} D_i \mathcal{Z} = \alpha \epsilon^i F_{\mathcal{Z}} \mathcal{Z} \overline{D}_i \mathcal{Z} \equiv \alpha \epsilon^i \overline{D}_i \int d\mathcal{Z} \; F_{\mathcal{Z}} \mathcal{Z} \; . \tag{7}$$

Thus, all terms in (6) are either full time derivatives or disappear after integration over $d^2\bar\theta$.

The irreducible component content of $\mathcal{Z}$, implied by (4), does not depend on α and can be defined as

$$z = \mathcal{Z}|, \; \bar z = \overline{\mathcal{Z}}|, \; A = -i\overline{D}^i \overline{D}_i \mathcal{Z}|, \; \bar A = -iD^i D_i \overline{\mathcal{Z}}|, \; \psi^i = \overline{D}^i \mathcal{Z}|, \; \bar\psi^i = -D^i \overline{\mathcal{Z}}|, \tag{8}$$

where $|$ means restricting expressions to $\theta_i = \bar\theta^j = 0$. All higher-dimensional components are expressed as time derivatives of the irreducible ones. Thus, the $N = 4$ superfield $\mathcal{Z}$ constrained by (4) has the same field content as the linear chiral supermultiplet.

Due to the nonlinearity of the basic constraints (4), the transformation properties of the components (8) also contain nonlinear terms[3]

$$\begin{aligned}
&\delta z = (\alpha \epsilon_i z + \bar\epsilon_i)\psi^i, \quad \delta\psi^i = \frac{i}{2}(\bar\epsilon^i + \alpha\epsilon^i z)A - 2i\epsilon^i \dot z + \frac{1}{2}\alpha\epsilon^i (\psi)^2, \quad \delta A = -4\epsilon^i \dot\psi_i, \\
&\delta \bar z = (\alpha \bar\epsilon^i \bar z - \epsilon^i)\bar\psi_i, \quad \delta\bar\psi_i = \frac{i}{2}(\epsilon_i - \alpha\bar\epsilon_i \bar z)\bar A + 2i\bar\epsilon_i \dot{\bar z} - \frac{1}{2}\alpha\bar\epsilon_i (\bar\psi)^2, \quad \delta \bar A = 4\bar\epsilon^i \dot{\bar\psi}_i.
\end{aligned} \tag{9}$$

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[2] We use the convention $\int d^2\theta = \frac{1}{4} D^i D_i$, $\int d^2\bar\theta = \frac{1}{4}\overline{D}_i \overline{D}^i$, $\int d^2\theta d^2\bar\theta = \frac{1}{16} D^i D_i \overline{D}_i \overline{D}^i$.

[3] All implicit summations go from "up-left" to "down-right", e.g., $\psi\bar\psi \equiv \psi^i \bar\psi_i$, $\psi^2 \equiv \psi^i \psi_i$, etc.

Components formulation.

After integrating in (5) over the Grassmann variables and eliminating the auxiliary fields $A, \bar{A}$ by their equations of motion, we get the action in terms of physical components

$$\begin{aligned} S \;\; = \;\; & \int dt \Big\{ g\dot{z}\dot{\bar{z}} - i\alpha\frac{\dot{z}\bar{z}}{1+\alpha^2 z\bar{z}}F_z + i\alpha\frac{\dot{\bar{z}}z}{1+\alpha^2 z\bar{z}}\overline{F}_{\bar{z}} - \frac{F_z\overline{F}_{\bar{z}}}{g(1+\alpha^2 z\bar{z})^2} + \\ & \frac{i}{4}(1+\alpha^2 z\bar{z})g\left[\psi^i\dot{\bar{\psi}}_i - \dot{\psi}^i\bar{\psi}_i + \psi^i\bar{\psi}_i\left(\frac{g_{\bar{z}}}{g}\dot{\bar{z}} - \frac{g_z}{g}\dot{z} + \alpha^2\frac{z\dot{\bar{z}} - \dot{z}\bar{z}}{(1+\alpha^2 z\bar{z})}\right) + \alpha\frac{\dot{\bar{z}}\psi^2 + \dot{z}\bar{\psi}^2}{(1+\alpha^2 z\bar{z})}\right] + \\ & \frac{1}{4}(\psi)^2\Big[\frac{2\alpha^2\bar{z}F_z}{(1+\alpha^2 z\bar{z})} - F_{zz} + F_z\frac{g_z}{g}\Big] - \frac{1}{4}(\bar{\psi})^2\Big[\frac{2\alpha^2 z\overline{F}_{\bar{z}}}{(1+\alpha^2 z\bar{z})} - \overline{F}_{\bar{z}\bar{z}} + \overline{F}_{\bar{z}}\frac{g_{\bar{z}}}{g}\Big] - \\ & \frac{1}{16}(\psi)^2(\bar{\psi})^2\Big[2\alpha^2 g + (1+\alpha^2 z\bar{z})^2 g_{z\bar{z}} - (1+\alpha^2 z\bar{z})^2\frac{g_z g_{\bar{z}}}{g}\Big]\Big\}, \end{aligned} \tag{10}$$

where

$$g(z,\bar{z}) = \frac{\partial^2 K(z,\bar{z})}{\partial z\partial\bar{z}}, \quad F_z = \frac{dF(z)}{dz}, \quad \overline{F}_{\bar{z}} = \frac{d\overline{F}(\bar{z})}{d\bar{z}} \ . \tag{11}$$

Using the Noether theorem one can find classical expressions for the conserved supercharges corresponding to the supersymmetry transformations (9)

$$\begin{aligned} Q^i &= g\dot{\bar{z}}\psi^i - \alpha g\dot{z}\bar{z}\bar{\psi}^i - \frac{i}{4}\alpha^2 gz(\psi)^2\bar{\psi}^i + \frac{i}{4}\alpha g(\bar{\psi})^2\psi^i - i\frac{\alpha\bar{z}F_z}{1+\alpha^2 z\bar{z}}\psi^i + i\frac{\overline{F}_{\bar{z}}}{1+\alpha^2 z\bar{z}}\bar{\psi}^i \ , \\ \overline{Q}_i &= g\dot{z}\bar{\psi}_i + \alpha g\dot{\bar{z}}z\psi_i + \frac{i}{4}\alpha^2 zg(\bar{\psi})^2\psi_i + \frac{i}{4}\alpha g(\psi)^2\bar{\psi}_i + i\frac{\alpha z\overline{F}_{\bar{z}}}{1+\alpha^2 z\bar{z}}\bar{\psi}_i + i\frac{F_z}{1+\alpha^2 z\bar{z}}\psi_i \ . \end{aligned} \tag{12}$$

From the bosonic part of the action, given by the first line in (10), one may conclude that the system contains a *nonzero magnetic field* with the potential

$$\mathcal{A}_0 = i\alpha\frac{F_z\bar{z}dz}{1+\alpha^2 z\bar{z}} - i\alpha\frac{\overline{F}_{\bar{z}}zd\bar{z}}{1+\alpha^2 z\bar{z}} \ , \ d\mathcal{A}_0 = i\alpha\frac{F_z + \overline{F}_{\bar{z}}}{(1+\alpha^2 z\bar{z})^2}dz\wedge d\bar{z}. \tag{13}$$

The strength of this magnetic field is given by the expression

$$B = \alpha\frac{(F_z + \overline{F}_{\bar{z}})}{(1+\alpha^2 z\bar{z})^2 g} \ . \tag{14}$$

The bosonic potential is also modified

$$V(z,\bar{z}) = \frac{F_z\overline{F}_{\bar{z}}}{(1+\alpha^2 z\bar{z})^2 g}. \tag{15}$$

One could represent the fermionic part of the kinetic term as follows:

$$\mathcal{S}_{KinF} = \frac{i}{4}\int dt(1+\alpha^2 z\bar{z})g\left(\psi\frac{D\bar{\psi}}{dt} - \bar{\psi}\frac{D\psi}{dt}\right), \tag{16}$$

where

$$D\psi = d\psi + \Gamma\psi dz + T^+\bar{\psi}dz, \quad D\bar{\psi} = d\bar{\psi} + \bar{\Gamma}\bar{\psi}d\bar{z} + T^-\psi d\bar{z}, \tag{17}$$

and

$$\Gamma = \partial_z \log\left((1+\alpha^2 z\bar{z})g\right), \quad T^\pm = \pm\frac{\alpha}{1+\alpha^2 z\bar{z}}. \tag{18}$$

Clearly enough, Γ, $\bar{\Gamma}$, $T^\pm$ define the components of the connection defining the configuration superspace. The components Γ and $\bar{\Gamma}$ could be identified with the components of the symmetric connection on the base space equipped with the metric $(1+\alpha^2 z\bar{z})gdzd\bar{z}$, while the rest does not have a similar interpretation.

Thus, we conclude that the main differences between the proposed $N=4$ supersymmetric mechanics with NCS and the standard one is the coupling of the fermionic degrees of freedom to the background, via the deformed connection, the possibility to introduce a magnetic field, and the deformation of the bosonic potential.

Landau problem

There is a distinguished case, when the underlying space is the sphere with $g = (1 + \alpha^2 z\overline{z})^{-2}$. Choosing $F'(z) = B_0/2 + ic_0$, we get a free particle on the sphere in a constant magnetic field of magnitude B_0, and a trivial (constant) potential $V_0 = (B_0/2)^2 + c_0^2$, i.e. the Landau problem on the sphere. Notice that very recently the simplest supersymmetric extention of the Landau problem was used for developing the theory of supersymmetric quantum Hall effect [7]. Our model gives the $N = 4$ extended supersymmetric background for that theory.

For $F'(z) = \omega_0 z$ we get the potential for the oscillator on $CP^1 = S^2$ (which is an exactly solvable system, even in the presence of a constant uniform magnetic field) [8], in the non-uniform magnetic field $B = \alpha\omega_0(z + \bar{z})$.

One can observe that similar systems on the Lobachevsky plane ($g = (1 - z\bar{z})^{-2}$, $\alpha = 1$) behave very differently from the ones on the sphere.

With the previously choosen superpotential $F'(z) = B_0/2 + ic_0$, we get the potential of the Higgs oscillator on the Lobachevsky plane (in the absence of a magnetic field this system is not only exactly solvable, but also "maximally integrable") $V = \left((B_0/2)^2 + c_0^2\right)\left(1 + 4z\bar{z}(1 + z\bar{z})^{-2}\right)$ [9] and the non-uniform magnetic field $B = B_0\left(1 + 2z\bar{z}(1 + z\bar{z})^{-2}\right)$. It seems obvious that for $B_0 = 0$ the corresponding $N = 4$ supersymmetric system will be exactly solvable as well. For $F'(z) = \omega_0 z$ the potential and the magnetic field are given by the expressions $V_0 = \omega_0^2 z\bar{z}(1 - 4z\bar{z}(1 + z\bar{z})^{-2})$, $B = \omega(z + \bar{z})(1 - 4z\bar{z}(1 + z\bar{z})^{-2})$.

This asymmetry between the systems on the sphere and the Lobachevsky plane is not so surprising, if we recall that the system was built by using the chiral supermultiplet constructed on the coset $S^2 = SU(2)/U(1)$, rather than $SU(1,1)/U(1)$.

Conclusion

In the present paper we constructed $N = 4$ supersymmetric mechanics with the nonlinear chiral supermultiplet. The main interesting peculiarities of constructed system are the non-standard coupling of the fermionic sector to the background, the appearance in the action of the interaction with the magnetic field having the strength

$$B = \alpha\frac{(F_z + \overline{F}_{\bar{z}})}{(1 + \alpha^2 z\overline{z})^2 g},$$

and the deformation of the bosonic potential

$$\frac{F_z\overline{F}_{\bar{z}}}{g} \to \frac{F_z\overline{F}_{\bar{z}}}{(1 + \alpha^2 z\overline{z})^2 g}.$$

Let us recall that usually the appearance of a magnetic field breaks supersymmetry, although it preserves the form of the bosonic part of the potential. Here, we have the opposite situation. This allows us to include in the class of $N = 4$ supersymmetrizeable systems the Landau problem on the sphere and the Higgs oscillator on the Lobachewski plane.

These results should be regarded as preparatory for more detailed study of supersymmetric mechanics with nonlinear supermultiplets. One of the obvious questions is the quantization of the system, and the construction of its $2n$-dimensional generalization corresponding to the dependence of the Lagrangian on n nonlinear chiral multiplets. It is interesting to get such a nonlinear analog for $N = 8$ supersymmetric mechanics (on special Kähler manifolds) [10] and for duality transformations of such a system. Also, it is still unclear whether is it possible to extend the system to higher space-time dimensions. Finally, let us recall that in one dimension there is the possibility to turn the auxiliary bosonic variables into dynamical ones [2]. Similarly, one may convert dynamical variables into auxiliary ones. Until now, such a procedure has been applied only to linear supermultiplets. It is interesting to check whether it works for nonlinear supermultiplets, as well.

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